

Rigged Hilbert Space Approach to Spectral Analysis of the Frobenius–Perron Operator for the Tent–map

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On the basis of an unified theoretical formulation of resonances and resonance states in the rigged Hilbert spaces the spectral analysis of the Frobenius–Perron operators corresponding to the exactly solvable chaotic map has been developed. Tent map as the simplest representative of exactly solvable chaos have been studied in details in frames of the developed approach. An extension of the Frobenius–Perron operator resolvent to a suitable rigged Hilbert space has been constructed in particular and the properties of the generalized spectral decomposition have been studied. Resonances and resonance projections for this map have been calculated explicitly.

1 Introduction

The statistical description of the dynamical systems in terms of ensembles have been introduced by Gibbs and Einstein. It allowed to calculate "suitably" the averages and, what is more important, the notion of the ensembles proved necessary for the description of the thermodynamical equilibrium. In general, one can understand the thermodynamical properties systems in terms of ensembles but not in the terms of the trajectories [8], [24]. So, the distribution of the probabilities becomes the main value. Then the question appears: to which functional space the probability should belong. E. g. for the Bernoulli shift one can construct the spectral representation in the Hilbert space $L_2(0, 1)$ [9]. But the spectrum of the Bernoulli shift which is isomorphic to the unilateral shift [6] is not linked in $L_2(0, 1)$ with Lyapunov time and, so, does not allow to describe the approach to equilibrium. If we examine the Frobenius–Perron operator [7], [15] for the Bernoulli shift not in $L_2(0, 1)$ but in the rigged Hilbert space $\Phi \subset L_2 \subset \Phi^\times$ one should change the notion of the spectral representation [10], [14]. The structure of the rigged Hilbert space $\Phi \subset \mathcal{H} \subset \Phi^\times$ allows to include into the spectral decomposition the strong irregular functions which do not belong to $L_2(0, 1)$. In the papers on the spectral analysis of the Frobenius–Perron operator in the rigged spaces [1], [4], [18] it was noted that the "non-Hilbert" spectrum is linked directly with Lyapunov time and so characterizes the temporary horizon of the chaotic systems.

It should be noted that the trajectories for the Bernoulli shift can be represented by the delta-function $\delta(x - x_n)$ where $x_n = S^n x_0$ and, formally, the Frobenius–Perron operator for the Bernoulli shift can act on the delta-function too. Indeed, since the action to the densities is

given by the formula [6]

$$(U\rho)(x) = \frac{1}{2} \left[\rho\left(\frac{x}{2}\right) + \rho\left(1 + \frac{x}{2}\right) \right] \quad (1)$$

so $U : \delta(x - x_n) \rightarrow \delta(x - 2x_n)$ at $x_n \leq 1/2$ and $U : \delta(x - x_n) \rightarrow \delta(x + 1 - 2x_n)$ at $1/2 \leq x \leq 1$. But the spectral representation in the rigged Hilbert space $\Phi \subset \mathcal{H} \subset \Phi^\times$

$$U = \sum_n \frac{1}{2} |B_n\rangle \langle \tilde{B}_n| \quad (2)$$

is applicable only to smooth densities $\rho \in \Phi$. It is because the left eigenvector $\langle \tilde{B}_n|$ is some functional of delta-like type. So, the spectral representation for U in $\Phi \subset \mathcal{H} \subset \Phi^\times$ is correct only for the smooth set of the trajectories but not for the individual trajectory separately. It is the fundamental result which states that for the chaotic systems the description on the language of the distributions can not be reduced to the description on the terms of the trajectories. It is the principal difference of the approach based on the spectral representation in rigged Hilbert space from the theory of the Gibbs–Einstein.

In general, one should use the different approaches for constructing the spectral representation for one or another class of the operators. So, e. g. for the self-adjoint, unitary or normal operators the classical spectral theorem works good. For the operators more "complicated" than the above-mentioned something more suitable should be developed. For the arbitrary compact operators, in particular, the spectral theorem can not give the exhaustive description [11]. The typical contr-example is the operator $(T\rho)(x) = \int_0^x \rho(x') dx'$ in $L_2(0, 1)$. The operator T has no eigenvalues and its spectrum is localized at zero: $\sigma(T) = \{0\}$. The reconstruction of the spectral theory of such type of operators needs approaches different from the spectral theorem [26].

In present paper we use the approach of the rigged Hilbert spaces [13], [14] and construct the generalized spectral decomposition of the Frobenius–Perron operator for the tent-map based on the extensions of its resolvent. This approach has been proposed in [1] and applied, in particular, for constructing of the spectral analysis of the Frobenius–Perron operator for the Renyi map. In [1], [4] it has been noted that the generalized spectrum for this map consists of the points $z_n = \beta^{-n}$, $n \geq 1$, $\beta \geq 2$ and the left and right eigenfunctions $\langle \tilde{\psi}_n|$ and $|\psi_n\rangle$ belong to Φ^\times and Φ respectively. So, the different modes of $\rho(x)$ approach to the equilibrium distribution $\rho_*(x)$ with the characteristic Lyapunov times $\gamma_n = n \ln \beta$. This fact follows directly from the spectral decomposition for the corresponding Frobenius–Perron operator [1], [4]:

$$(U_{RP}^t \rho)(x) = \sum_{n \geq 0} e^{-\gamma_n t} |\psi_n\rangle \langle \tilde{\psi}_n| \rho, \quad t = 1, 2, 3 \dots \quad (3)$$

As it mentioned above we study the dynamic of the chaotic tent map

$$S : x \rightarrow \begin{cases} 2x, & 0 \leq x \leq 1/2 \\ 2 - 2x, & 1/2 \leq x \leq 1 \end{cases} \quad (4)$$

on the base of the spectral representation of its Frobenius–Perron operator U_T in the suitable rigged space $\Phi \subset L_2 \subset \Phi^\times$. So, instead of studying individual trajectories, generating by the iterations of the map S : $x_n = S^n x_0$, $x_0 \in [0, 1]$ we use an approach characteristic for the nonequilibrium statistical mechanics [12], i.e. we observe the evolution in the time of the densities $\rho(x) \in \Phi$. The evolution of the latter is given by the operator U_T^t , $t = 1, 2, 3 \dots$, i.e. $\rho(x, t) = U_T^t \rho_0(x)$, where $\rho_0(x)$ is an initial density of the distribution. In our approach the rate of the decay of the initial density is determined by the non-Hilbert spectrum $\sigma(U_T^t)$

of the Frobenius–Perron operator. In frames of the approach developed in [21] the rates of the decay correspond to the poles of the Fourier transformation of the correlation function and so, are interpreted as the resonances of the dynamical system. We note that for some dynamical systems these poles has been studied in [27], [28], [29] on the basis of the theory of the periodical orbits and dynamical ξ -function [19], [20]. So, the generalized spectrum in our approach coincides with what is known as Ruelle–Pollicott resonances.

The detailed operator construction, using in this work, is described briefly in following section. Here we note only that our approach is similar to the methods of [4], [1], [16], [22] and is different essentially from the technique used e.g. in [19].

2 Spectral decomposition in rigged Hilbert spaces

In this section we introduce necessary notions and describe briefly the approach based on the extensions of the Frobenius–Perron operator for one-dimensional maps.

Let $S : [0, 1] \rightarrow [0, 1]$ be one-dimensional map, $x \in [0, 1]$. Denote as $S^{-1}([0, x]) = \{y : S(y) \in [0, x]\}$ and let $\rho(x)$ be a density on $[0, 1]$. We suppose that $\rho(x)$ is so, that all integrals below exist. Then the Frobenius–Perron operator U associated with the map S is given by the formula [7], [15]

$$\int_0^x U\rho(x') dx' = \int_{S^{-1}([0, x])} \rho(x') dx'. \quad (5)$$

The Koopman operator V for the map S can be defined as [7], [15]

$$(V\rho)(x) = \rho(Sx). \quad (6)$$

We shall start with the densities ρ such that

$$\rho \in \mathcal{H} = L^2(0, 1). \quad (7)$$

Then, denoting inner product in \mathcal{H} as $\langle \cdot, \cdot \rangle$ using (5), (6) one can show that for any functions f and g from \mathcal{H} ,

$$\langle Uf, g \rangle = \langle f, Vg \rangle. \quad (8)$$

So, the Koopman operator V is adjoint to Frobenius–Perron operator U , i. e.

$$V = U^*. \quad (9)$$

Equations (8) and (9) give a possibility to calculate U if we know V and vice versa. For the tent-map one can obtain

$$S^{-1}([0, x]) = \left[0, \frac{x}{2}\right] \cup \left[1 - \frac{x}{2}, 1\right] \quad (10)$$

and

$$\int_0^x U_T\rho(x') dx' = \int_0^{x/2} \rho(x') dx' + \int_{1-x/2}^1 \rho(x') dx'. \quad (11)$$

It means that U_T reads as

$$(U_T\rho)(x) = \frac{1}{2} \left[\rho\left(\frac{x}{2}\right) + \rho\left(1 - \frac{x}{2}\right) \right]. \quad (12)$$

On the other hand, according to (8), (4) one have

$$\begin{aligned}
\langle V_T f, g \rangle &= \int_0^1 f(Sx) \overline{g(x)} dx = \int_0^{1/2} f(2x) \overline{g(x)} dx + \int_{1/2}^1 f(2(1-x)) \overline{g(x)} dx = \\
&= \frac{1}{2} \int_0^1 f(x) g\left(\frac{x}{2}\right) dx + \frac{1}{2} \int_0^{1/2} f(x) g\left(1 - \frac{x}{2}\right) dx = \\
&= \int_0^1 f(x) \frac{1}{2} \left[g\left(\frac{x}{2}\right) + g\left(1 - \frac{x}{2}\right) \right] dx = \langle f, V_T^* \rangle = \langle f, U_T g \rangle,
\end{aligned}$$

that leads again to the equation (12).

In terms of Frobenius–Perron operator it is possible to determine the stationary (equilibrium) density $\rho_*(x)$ and corresponding invariant measure $\mu_*(x)$. Namely $\rho_*(x)$ is determined by the equation

$$U\rho_* = \rho_*, \quad (13)$$

and $\mu_*(x)$ is given by the formula

$$\mu_*(x) = \int_0^x \rho_*(x') dx'.$$

Solving the functional equation for the equilibrium density ρ_*

$$\rho_*(x) = \frac{1}{2} \left[\rho_*\left(\frac{x}{2}\right) + \rho_*\left(1 - \frac{x}{2}\right) \right] \quad (14)$$

one can find that

$$\rho_*(x) = 1 \quad (15)$$

and hence

$$\mu_*(x) = x. \quad (16)$$

So the invariant measure of the tent-map is the Lebegues measure.

We shall study the operators U_T and V_T in the rigged Hilbert spaces and in this connection, we shall describe the scheme of the spectral analysis in these spaces [1] below.

The space Φ is a test space for the operator U_T if:

1) Φ is a locally convex topological vector space with topology stronger than the Hilbert space topology \mathcal{H} ;

2) Φ is continuously and densely embedded in \mathcal{H} ;

3) Φ is invariant to the adjoint operator U_T^* : i. e. to V_T : $U_T^* \Phi \subset \Phi$.

The dual space Φ^\times to Φ we shall consider as the linear continuous functionals on Φ . It is clear, that the topology in Φ^\times is weaker than in \mathcal{H} . The space $\Phi \subset \mathcal{H} \subset \Phi^\times$ we shall call rigged Hilbert space or Gelfand triplet. The coupling in $\Phi \subset \mathcal{H} \subset \Phi^\times$ we shall denote as $\langle \cdot | \cdot \rangle$.

The operator \tilde{U}_T is called an extension of the operator U_T to the dual space Φ^\times if \tilde{U}_T acts on linear functionals $\langle f | \in \Phi^\times$ as:

$$\langle \tilde{U}_T f | \varphi \rangle = \langle f | U_T^* \varphi \rangle \quad (17)$$

for all test functions $\varphi \in \Phi$.

We define the extended resolvent $\tilde{R}_U(z)$ to a rigged Hilbert space as follows [1]: we shall call the operator-valued function $\tilde{R}_U(z)$ an extended resolvent of the operator U in a suitable rigged Hilbert space $\Phi \subset \mathcal{H} \subset \Phi^\times$ if:

1) $\tilde{R}_U(z)$ satisfies the resolvent identity in the weak sense:

$$\langle \varphi | \tilde{R}_U(z) (U_T - z) | \psi \rangle = \langle \varphi | \psi \rangle, \quad \forall \varphi, \psi \in \Phi, \quad (18)$$

where z is not a singular point of the $\tilde{R}_U(z)$;

2) $\tilde{R}_U(z)$ satisfies a completeness condition in a weak sense, i. e.

$$-\frac{1}{2\pi i} \oint_{\Gamma} \langle \varphi | \tilde{R}_U(z) | \psi \rangle dz = \langle \varphi | \psi \rangle, \quad \forall \varphi, \psi \in \Phi, \quad (19)$$

where the contour Γ encircles all the singularities of the integrand in (19) in the positive direction.

Then, from (18) the dependence between the extended operator \tilde{U}_T and the extended resolvent $\tilde{R}_U(z)$, is, as usual,

$$\tilde{R}_U(z) = (\tilde{U}_T - z\mathbf{I})^{-1}. \quad (20)$$

So the extended operator \tilde{U}_T corresponds to the extended resolvent $\tilde{R}_U(z)$.

We define the Gelfand spectrum $\hat{\sigma}(\tilde{U}_T)$ of the operator \tilde{U}_T as a set of singularities of the extended resolvent $\tilde{R}_U(z)$ on the complex plane $z \in \mathbf{C}$. We shall not give here the detailed classification of the spectral components of $\hat{\sigma}(\tilde{U}_T)$ and restrict ourselves by the case when all singularities of the extended resolvent $\tilde{R}_U(z)$ are the poles: $\hat{\sigma}(\tilde{U}_T) = \{z_\nu\}$. The residues in the poles $\{z_\nu\}$ of $\tilde{R}_U(z)$ will be referred as the generalized spectral projections of the operator \tilde{U}_T :

$$\Pi_\nu = \text{res} \tilde{R}_U(z)|_{z=z_\nu} = -\frac{1}{2\pi i} \oint_{C_\nu} \tilde{R}_U(\zeta) d\zeta. \quad (21)$$

For the case of simple poles Π_ν is one-dimensional projection:

$$\Pi_\nu = |\psi_\nu\rangle \langle \tilde{\psi}_\nu|, \quad (22)$$

where $|\psi_\nu\rangle$ and $\langle \tilde{\psi}_\nu|$ are the generalized right and left eigenvectors of the corresponding extended operator \tilde{U}_T :

$$\begin{aligned} \tilde{U}_T |\psi_\nu\rangle &= z_\nu |\psi_\nu\rangle, \\ \langle \tilde{\psi}_\nu | \tilde{U}_T &= z_\nu \langle \tilde{\psi}_\nu|. \end{aligned} \quad (23)$$

Then, from the decomposition (19) it follows that the generalized spectral decomposition for \tilde{U}_T looks like

$$\tilde{U}_T = \sum_{\nu} z_\nu \Pi_\nu. \quad (24)$$

The latter is considered as decomposition of density $\rho \in \Phi$ by the (biorthogonal) system of eigenfunctions $\{|\psi_\nu\rangle, \langle \tilde{\psi}_\nu|\}$:

$$\tilde{U}_T \rho = \sum_{\nu} z_\nu |\psi_\nu\rangle \langle \tilde{\psi}_\nu | \rho. \quad (25)$$

The described construction can be used also for the extension of the Koopman operator \tilde{V}_T [1], [16].

Summarizing we can formulate the following algorithm for the construction of the spectral analysis of the Frobenius–Perron operator in the triplet $\Phi \subset \mathcal{H} \subset \Phi^\times$:

1. Fixe the test space Φ in \mathcal{H} and study the restriction $R_U(z)|_\Phi$ of the resolvent of the operator U_T .

2. Make the extension of the resolvent $R_U(z)|_\Phi$ to $\tilde{R}_U(z)$ and find all singularities $\tilde{R}_U(z)$ in the complex plane $z \in \mathbf{C}$.

3. Construct the decomposition of the $\tilde{R}_U(z)$ in weak sense, i. e. verify the conditions of the completeness (19).

4. If all singularities of $\tilde{R}_U(z)$ are the poles, then calculate the generalized spectral projections Π_ν .

5. Study the convergence of the generalized spectral decomposition (24) in the sense of (25).

In the following sections this algorithm shall be realized for the Frobenius–Perron operator corresponding to the tent map.

3 The resolvent of Frobenius–Perron operator in the Fourier representation

Let $\mathcal{A}[0, 1]$ be the algebra of analytic functions on the interval $[0, 1]$. It is known that $\mathcal{A}[0, 1]$ is continuously and densely embedded into the space $L_2[0, 1]$ and so one can consider the restriction of the Frobenius–Perron operator U_T on $\mathcal{A}[0, 1]$ choosing the latter as the space Φ of the test functions in the Gelfand triplet $\Phi \subset L_2[0, 1] \subset \Phi^\times$.

Let F be the Fourier transform on $\mathcal{A}[0, 1]$. Then,

$$F : \mathcal{A}[0, 1] \longrightarrow \widehat{\mathcal{A}[0, 1]} \subset l_2, \quad (26)$$

$$F\rho(x) = \hat{\rho} = \{\rho_n\}_{n=-\infty}^{n=+\infty} \in l_2, \quad (27)$$

$$\rho(x) = \sum_{n=-\infty}^{\infty} \rho_n e^{2i\pi nx}, \quad \rho_n = \int_0^1 \rho(x) e^{-2i\pi nx} dx, \quad (28)$$

and the series in (28) converges in L_2 -norm.

Let us calculate the action of the operator $\hat{U}_T = FU_TF^{-1}$:

$$\begin{aligned} (U_T\rho)(x) &= \sum_{n=-\infty}^{\infty} \rho_n U_T e^{2i\pi nx} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \rho_n (e^{i\pi nx} + e^{-i\pi nx}) = \\ &= \frac{1}{2} \sum_{n=-\infty}^{\infty} (\rho_n + \rho_{-n}) e^{i\pi nx}. \end{aligned}$$

Since

$$(U_T\rho)(x) = \sum_{m=-\infty}^{\infty} (\hat{U}\rho)_m e^{2i\pi mx},$$

the coefficients of these series are given by the expression

$$(\hat{U}_T\rho)_m = \int_0^1 (\hat{U}_T\rho)(x) e^{-2i\pi mx} dx = \frac{1}{2} \sum_{n=-\infty}^{\infty} (\rho_n + \rho_{-n}) \int_0^1 e^{i\pi(n-2m)x} dx.$$

Dividing the latter series into two series with summation by even and odd indices we have

$$\begin{aligned} (\hat{U}_T\rho)_m &= \frac{1}{2} \left(\sum_k (\rho_{2k} + \rho_{-2k}) \int_0^1 e^{i\pi(2k-2m)x} dx \right. \\ &\quad \left. + \sum_k (\rho_{2k+1} + \rho_{-(2k+1)}) \int_0^1 e^{i\pi(2k+1-2m)x} dx \right). \end{aligned}$$

The first integral in the right hand side of this expression equals to δ_{km} . So we have

$$\left(\hat{U}_T \rho\right)_m = \frac{1}{2}(\rho_{2m} + \rho_{-2m}) + \frac{i}{\pi} \sum_k \frac{\rho_{2k+1} + \rho_{-(2k+1)}}{2k - 2m + 1} \quad (29)$$

where we used the relation

$$\int_0^1 e^{i\pi(2k+1-2m)x} dx = \frac{2i}{\pi} \cdot \frac{1}{2k + 1 - 2m}.$$

The representation(29) can be used for finding the solution of the equation

$$\left((\hat{U}_T - z)\rho\right)_m = f_m, \quad \{f_m\}_{m=-\infty}^{m=+\infty} \in l_2, \quad (30)$$

which is the base for obtaining the resolvent $\hat{R}_T(z) = (\hat{U}_T - z)^{-1}$ of the operator \hat{U}_T . Indeed, in an explicit form (30) reads for $m \neq 0$ as

$$\rho_m = \frac{1}{z} \left[\frac{1}{2}(\rho_{2m} + \rho_{-2m}) + \frac{i}{\pi} \sum_k \frac{\rho_{2k+1} + \rho_{-(2k+1)}}{2k - 2m + 1} \right] - \frac{1}{z} f_m, \quad (31)$$

and for $m = 0$ as

$$\rho_0 = \frac{1}{1 - z} f_0. \quad (32)$$

It is possible to solve (31) by iteration, as it was done in [1], [4] for the Renyi map. But the iteration procedure for the equation (31) turns out to be very cumbersome. Therefore we shall use an additional trick based on the Feshbach projection technique [2] and the properties of symmetry [3] of the map with respect to the point $x = 1/2$.

4 The solution of the equation for the resolvent by the Feshbach projection method

Let us define for all functions f , $f \in \mathcal{A}[0, 1]$ the operator R of the reflection with respect to the point $x = 1/2$ by the formula

$$Rf(x) = f(1 - x), \quad (33)$$

and in terms of R on $\mathcal{A}[0, 1]$ we define the pair of the operators

$$P^\pm = \frac{1}{2}(I \pm R).$$

Obviously, that the operators P^\pm satisfy the following equations:

$$\begin{aligned} P^+ + P^- &= I, \\ (P^\pm)^2 &= P^\pm, \\ (P^+)^* &= P^-, \\ P^+ P^- &= P^- P^+ = 0. \end{aligned} \quad (34)$$

which mean that P^+ , P^- is the pair of the orthoprojections in $\mathcal{A}[0, 1] \subset L_2(0, 1)$. Hence, using P^+ , P^- one can split $L_2(0, 1)$ into the orthogonal sum $L_2^+ \oplus L_2^-$ of the even and odd functions with respect to the point $x = 1/2$ respectively.

Following [1] we rewrite the equation for the resolvent

$$(U_T - z)\rho(x) = f(x)$$

in the form:

$$(U_T - z)(P^+ + P^-)\rho(x) = f(x). \quad (35)$$

Acting by P^+ from the left on (35) one can obtain according to (34)

$$P^+U_TP^+\rho - zP^+\rho + P^+U_TP^-\rho = P^+f. \quad (36)$$

Similarly, acting on (35) by the operator P^- from the left we get

$$P^-U_TP^-\rho - zP^-\rho + P^-U_TP^+\rho = P^-f \quad (37)$$

Denoting $P^\pm\rho = \rho^\pm$, $P^\pm f = f^\pm$, we obtain from (36), (37) the set of equations with respect to the vector $\rho^+ \oplus \rho^-$:

$$\begin{pmatrix} P^+U_TP^+ - z & P^+U_TP^- \\ P^-U_TP^+ & P^-U_TP^- - z \end{pmatrix} \cdot \begin{pmatrix} \rho^+ \\ \rho^- \end{pmatrix} = \begin{pmatrix} f^+ \\ f^- \end{pmatrix}. \quad (38)$$

Let us note that for the functions symmetrical with respect to the point $x = 1/2$

$$\rho_n = \rho_{-n} \quad \text{and} \quad \rho_n = -\rho_{-n}, \quad n \in \mathbf{Z} \quad (39)$$

for the functions antisymmetrical with respect to the point $x = 1/2$. Indeed, for the symmetrical functions $\rho(x)$:

$$\begin{aligned} \rho(1-x) &= \sum_{n=-\infty}^{\infty} \rho_n e^{2i\pi n(1-x)} = \sum_{n=-\infty}^{\infty} \rho_n e^{2i\pi n} e^{-2i\pi nx} = \\ &= \sum_{n=-\infty}^{\infty} \rho_{-n} e^{2i\pi nx} = \sum_{n=-\infty}^{\infty} \rho_n e^{2i\pi nx} = \rho(x). \end{aligned}$$

and analogously for the antisymmetrical functions $\rho(x)$. Then from (39), it follows, that

$$U_TP^-\rho^- = 0. \quad (40)$$

It means that for the solution of the system (38) in the Fourier representation only $P^+U_TP^+\rho^+$, and $P^-U_TP^+\rho^+$ blocks should be calculated. Namely:

$$\begin{aligned} (P^+U_TP^+)\rho^+ &= P^+U_T\rho^+ = P^+ \sum_{n=-\infty}^{\infty} \rho_n^+ e^{i\pi n x} = \sum_{n=-\infty}^{\infty} \rho_n^+ \frac{1+R}{2} e^{i\pi n x} = \\ &= \sum_{n=-\infty}^{\infty} \rho_n^+ \frac{e^{i\pi n x} + e^{i\pi n} e^{-i\pi n x}}{2} = \sum_{n=-\infty}^{\infty} \rho_n^+ \frac{e^{i\pi n x}}{2} + \sum_{n=-\infty}^{\infty} \rho_{-n}^+ \frac{e^{-i\pi n} e^{i\pi n x}}{2} = \\ &= \sum_{n=-\infty}^{\infty} \rho_n^+ \frac{1+e^{i\pi n}}{2} e^{i\pi n x}. \end{aligned}$$

Since $1 + e^{-i\pi n} = 0$ for $n = 2m + 1$, then

$$(P^+U_TP^+)\rho^+ = \sum_{m=-\infty}^{\infty} \rho_{2m}^+ e^{2i\pi m x},$$

and we find

$$\left(P^+ \hat{U}_T P^+ \rho^+\right)_m = \rho_{2m}^+ \quad m \in \mathbf{Z}. \quad (41)$$

In the same manner

$$\begin{aligned} (P^- U_T P^+) \rho^+ &= P^- U_T \rho^+ = P^- \sum_{n=-\infty}^{\infty} \rho_n^+ e^{i\pi n x} = \sum_{n=-\infty}^{\infty} \rho_n^+ \frac{1-R}{2} e^{i\pi n x} = \\ &= \sum_{n=-\infty}^{\infty} \rho_n^+ \frac{e^{i\pi n x} - e^{i\pi n} e^{-i\pi n x}}{2} = \sum_{n=-\infty}^{\infty} \rho_n^+ \frac{e^{i\pi n x}}{2} - \sum_{n=-\infty}^{\infty} \rho_{-n}^+ \frac{e^{-i\pi n} e^{i\pi n x}}{2} = \\ &= \sum_{n=-\infty}^{\infty} \rho_n^+ \frac{1 - e^{i\pi n}}{2} e^{i\pi n x}. \end{aligned}$$

Since $1 - e^{-i\pi n} = 0$ for $n = 2m$, so

$$(P^- U_T P^+) \rho^+ = \sum_{m=-\infty}^{\infty} \rho_{2m+1}^+ e^{i\pi(2m+1)x}.$$

Hence,

$$\left(P^- \hat{U}_T P^+ \rho^+\right)_l = \sum_{m=-\infty}^{\infty} \rho_{2m+1}^+ \int_0^1 e^{i\pi(2m+1)x - 2il\pi x} dx = \sum_{m=-\infty}^{\infty} \rho_{2m+1}^+ \frac{i}{\pi(2m+1-2l)}. \quad (42)$$

So the set of equations (38) in the Fourier representation gets the form (for $m \neq 0$):

$$\begin{cases} \rho_{2m}^+ - z \rho_m^+ &= f_m^+ \\ \sum_{l=-\infty}^{\infty} \rho_{2l+1}^+ \frac{2i}{\pi(2l-2m+1)} - z \rho_m^- &= f_m^-. \end{cases} \quad (43)$$

For the case $m = 0$ the solution ρ_0 is given by the formula (32).

Let us note that in (43) only the first equation for ρ_m^+ is essential because if ρ_m^+ is founded, then ρ_m^- is reconstructed from the second equation of the set:

$$\rho_m^- = \frac{1}{z} \sum_{l=-\infty}^{\infty} \rho_{2l+1}^+ \frac{i}{\pi(2l-2m+1)} - \frac{1}{z} f_m^-. \quad (44)$$

On the other hand, the first equation of the set (43)

$$\rho_{2m}^+ - z \rho_m^+ = f_m^+ \quad (45)$$

coincides with the equation for the resolvent of the Frobenius–Perron operator corresponding to the Bernoulli shift [1] and its solution is given by the formula (for $m \neq 0$)

$$\rho_m^+ = -\frac{1}{z} \sum_{k=0}^{\infty} z^{-k} f_{2^k m}^+. \quad (46)$$

The series in (46) converges absolutely if $|z| > 1/2$, [1] and $\hat{f} \in \hat{\mathcal{A}}[0, 1] \subset l_2$. In terms of ρ_m^+ one can reconstruct ρ_m^+ :

$$\rho_m^- = \frac{1}{z^2} \sum_{l=-\infty}^{\infty} \frac{i}{\pi(2l-2m+1)} \sum_{k=0}^{\infty} z^{-k} f_{(2l+1)2^k}^+ - \frac{1}{z} f_m^-. \quad (47)$$

Taking into account (40) we can rewrite (38) as

$$\begin{cases} (P^+ U_T P^+ - z) \rho^+ = f^+ \\ P^- U_T P^+ \rho^+ - z \rho^- = f^- \end{cases}$$

Then

$$\begin{cases} \rho^+ = (P^+ U_T P^+ - z)^{-1} f^+ \\ \rho^- = \frac{1}{z} (P^- U_T P^+ \rho^+) - \frac{1}{z} f^- \end{cases}$$

Hence,

$$\begin{aligned} \rho^+ + \rho^- &= P^+ \rho + P^- \rho = \rho = \\ &= \left[(P^+ U_T P^+ - z)^{-1} P^+ + \frac{1}{z} P^- U_T P^+ (P^+ U_T P^+ - z)^{-1} P^+ - \frac{1}{z} P^- \right] f. \end{aligned} \quad (48)$$

Comparing (48) with the presentation of the resolvent for the operator U_T :

$$\rho(x) = (U_T - z)^{-1} f,$$

we find that the complete resolvent $(U_T - z)^{-1}$ can be reconstructed from the partial resolvent $(P^+ U_T P^+ - z)^{-1}$ by the formula

$$(U_T - z)^{-1} = (P^+ U_T P^+ - z)^{-1} P^+ \frac{1}{z} P^- U_T P^+ (P^+ U_T P^+ - z)^{-1} P^+ - \frac{1}{z} P^-. \quad (49)$$

It is obvious that $f_n^\pm = \frac{1}{2}(f_n \pm f_{-n})$. Then from (46) and (47) we find for $n \neq 0$ the Fourier presentation for (49):

$$\rho_n = \rho_n^+ + \rho_n^- = -\frac{1}{z} \sum_{k=0}^{\infty} \frac{f_{n2k} + f_{-n2k}}{2z^k} - \frac{1}{z} \frac{f_n - f_{-n}}{2} - \frac{i}{2\pi z^2} \sum_{l=-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{f_{(2l+1)2k} + f_{-(2l+1)2k}}{z^k (2l - 2n + 1)}. \quad (50)$$

Let us show that both series in (50) converges absolutely in z if $|z| > 1/2$. Since $\rho(x) \in \mathcal{A}[0, 1]$ then integrating (28) by parts we obtain:

$$\begin{aligned} \rho_m &= \int_0^1 \rho(x) e^{-2i\pi m x} dx = \frac{i}{2\pi m} \left[\rho(x) \Big|_0^1 - \int_0^1 \rho'(x) e^{-2i\pi m x} dx \right] = \\ &= \frac{i}{2\pi m} \left[\rho(1) - \rho(0) - \int_0^1 \rho'(x) e^{-2i\pi m x} dx \right], \end{aligned} \quad (51)$$

where $\rho'(x)$ denotes the derivative. Denoting

$$c_\rho = (2\pi)^{-1} \left(|\rho(1) - \rho(0)| + \int_0^1 |\rho'(x)| dx \right), \quad (52)$$

we are convinced that $0 < c_\rho < \infty$ and

$$|\rho_m| \leq \frac{c_\rho}{m}. \quad (53)$$

Since $\hat{f} \in \widehat{\mathcal{A}[0, 1]}$ then analogously we obtain that

$$|f_m| \leq \frac{c_f}{m}$$

where c_f is some constant. It means that at $k \rightarrow \infty$ the following estimation

$$\left| \frac{1}{2z} f_{m2^k} \right| \leq c_f m^{-1} (2|z|)^{-k} \quad (54)$$

is valid. Hence the first series in (50) converges absolutely at

$$|z| > \frac{1}{2}. \quad (55)$$

Obviously, this condition is enough for the absolute convergence of the second series in (50).

Taking into account (50) we obtain the matrix elements $\hat{R}_{mn}(z)$ of the resolvent of the Frobenius–Perron operator U_T for the tent map as (for $n \neq 0$):

$$\begin{aligned} \hat{R}_{nm}(z) = & -\frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{z^{k+1}} (\delta_{2^k n, m} + \delta_{-2^k n, m}) - \frac{1}{2z} (\delta_{n, m} + \delta_{-n, m}) \\ & - \frac{1}{2} \sum_{k=0}^{\infty} \frac{i}{z^2 z^k \pi} \sum_{l=-\infty}^{\infty} \left[\frac{\delta_{(2l+1)2^k, m} + \delta_{-(2l+1)2^k, m}}{2l - 2n + 1} \right], \end{aligned} \quad (56)$$

and for $n = 0$:

$$\hat{R}_{00}(z) = \frac{1}{1-z} \quad (57)$$

From (56), (57) it follows that the resolvent $\hat{R}(z) = \{\hat{R}_{mn}\}$ is the meromorphic function in the domain $|z| > 1/2$ with a single simple pole at $z = 1$. Obviously the same is valid for the resolvent $R(z) = (U_T - z)^{-1} = F^{-1} \hat{R}(z) F$ in the space $\mathcal{A}[0, 1]$.

The residue of the resolvent $\hat{R}(z)$ at the point $z = 1$ is given by the integral

$$\text{res} \hat{R}(z)|_{z=1} = -\frac{1}{2i\pi} \oint_{\Gamma} \hat{R}(z) dz, \quad (58)$$

where the contour Γ is given by

$$\Gamma = \{z : z = 1 + r e^{i\varphi}, \quad r \in (0, 1/2), \quad \varphi \in [0, 2\pi)\}.$$

In virtue of the analyticity of $\hat{R}(z)$ only (57) gives the contribution in the residue. Then, denoting by $\tilde{\Pi}_0$ the operator with the matrix elements $\tilde{\Pi}_{nm} = \delta_{n,0} \delta_{m,0}$, we see that

$$\text{res} \hat{R}(z)|_{z=1} = \tilde{\Pi}_0. \quad (59)$$

Let us note, that introduced operator $\tilde{\Pi}_0$ is the projection on the vector \hat{q} , $q_n = \delta_{n,0}$. In the space $\mathcal{A}[0, 1]$ the corresponding operator $\Pi_0 = F^{-1} \tilde{\Pi}_0 F$,

$$F^{-1} \tilde{\Pi}_0 F = \text{res} R(z)|_{z=1} \quad (60)$$

has the kernel

$$\Pi_0(x, x') = \sum_{n,m} \tilde{\Pi}_{nm} e^{2i\pi n x} e^{-2i\pi m x'} = \mathbf{1}(x) \mathbf{1}(x'), \quad (61)$$

where $\mathbf{1}(x)$ is the function from $\mathcal{A}[0, 1]$ which identically equals to one on the interval $[0, 1]$.

Using bra- and ket- Dirac notions, the action of the operator $\tilde{\Pi}_0$ one can write formally as

$$\Pi_0 : \rho(x) \rightarrow (\Pi_0 \rho)(x) = \langle \mathbf{1}(x) | \rho(x) \rangle \mathbf{1}(x) = \int_0^1 \rho(x) dx \cdot \mathbf{1}(x). \quad (62)$$

5 The extension of the resolvent and the generalized spectral decomposition for the tent map

According to the general approach [1] one should extend the resolvent of the tent map outside its own domain of analytically $C_{1/2} = \{z : |z| > 1/2\}$ i.e. into the circle $C_{1/2}$. According to [1] the extension of the resolvent into this circle is possible in the topology weaker than Hilbert topology.

To build this extension, at first we calculate the kernel $R(x, x', z)$ of the resolvent $R(z) = (U_T - z)^{-1} = F^{-1} \hat{R}(z) F$ in the initial space $\mathcal{A}[0, 1]$. This kernel is given by

$$R(x, x', z) = \sum_{n,m} \hat{R}_{nm}(z) e^{2i\pi n x} e^{-2i\pi m x'} \quad (63)$$

and hence for all functions $\rho \in \mathcal{A}[0, 1]$ we have

$$(R(z)\rho)(x) = \int_0^1 R(x, x', z) \rho(x') dx' = \frac{1}{1-z} \int_0^1 \rho(x') dx' \mathbf{1}(x) + \Sigma_1 + \Sigma_2 + \Sigma_3, \quad (64)$$

where

$$\Sigma_1 = - \sum_{k \geq 0} \frac{1}{2z^{k+1}} \sum_{n \neq 0} e^{2i\pi n x} \left[\int_0^1 e^{-2i\pi n 2^k x'} \rho(x') dx' + \int_0^1 e^{2i\pi n 2^k x'} \rho(x') dx' \right], \quad (65)$$

$$\Sigma_2 = - \frac{1}{2z} \sum_{n \neq 0} e^{2i\pi n x} \left[\int_0^1 e^{-2i\pi n x'} \rho(x') dx' - \int_0^1 e^{2i\pi n x'} \rho(x') dx' \right], \quad (66)$$

$$\Sigma_3 = - \sum_{k \geq 0} \frac{i}{z^2 z^k \pi} \sum_{l,n} \frac{e^{2i\pi n x}}{2l - 2n + 1} \left[\int_0^1 e^{-2i\pi(2l+1)2^k x'} \rho(x') dx' + \int_0^1 e^{2i\pi(2l+1)n2^k x'} \rho(x') dx' \right]. \quad (67)$$

Let us calculate the series $\Sigma_1, \Sigma_2, \Sigma_3$ from (64) separately.

At first, let us consider the integral $\int_0^1 e^{2i\pi m x} \rho(x) dx$, $m \in \mathbf{Z}$. Then, integrating N times by parts, we have

$$\int_0^1 e^{2i\pi m x} \rho(x) dx = \sum_{s=1}^N \frac{(-1)^{s-1}}{(2i\pi m)^s} [\rho^{(s-1)}(1) - \rho^{(s-1)}(0)] + \frac{(-1)^N}{(2i\pi m)^{N-1}} \int_0^1 \rho^{(N)}(x) e^{2i\pi m x} dx. \quad (68)$$

We note now that in Σ_1 and Σ_3 one should integrate by parts the expression

$$\int_0^1 (e^{2i\pi m x} + e^{-2i\pi m x}) \rho(x) dx \quad (69)$$

where for Σ_1 $m = n2^k$ is integer and for Σ_3 $m = (2l+1)n2^k$ is integer. Then, similar to (68) we have

$$\begin{aligned} \int_0^1 (e^{2i\pi m x} + e^{-2i\pi m x}) \rho(x) dx &= \sum_{l=1}^N \frac{(-1)^{l-1}}{(2i\pi m)^l} [1 + (-1)^l] \cdot [\rho^{(l-1)}(1) - \rho^{(l-1)}(0)] \\ &+ \frac{(-1)^N}{(2i\pi m)^{N-1}} [1 + (-1)^{N-1}] \int_0^1 \rho^{(N)}(x) (e^{2i\pi m x} + e^{-2i\pi m x}) dx. \end{aligned} \quad (70)$$

In the right hand side of (70) the zero terms of the first sum correspond to odd l . Then, taking $l = 2s$ and replacing $N \rightarrow 2N + 1$ from (70) we obtain

$$\begin{aligned} \int_0^1 (e^{2i\pi mx} + e^{-2i\pi mx}) \rho(x) dx &= -2 \sum_{s=1}^N \frac{1}{(2i\pi m)^{2s}} [\rho^{(2s-1)}(1) - \rho^{(2s-1)}(0)] \\ &\quad - \frac{2}{(2i\pi m)^{2N}} \int_0^1 \rho^{(2N+1)}(x) (e^{2i\pi mx} + e^{-2i\pi mx}) dx. \end{aligned} \quad (71)$$

Let us note that in the expression Σ_2 there are the integrals of the type (69), but with the difference of its components. Consider such integral and integrating by parts we get

$$\begin{aligned} \int_0^1 (e^{2i\pi mx} - e^{-2i\pi mx}) \rho(x) dx &= \sum_{l=1}^N \frac{(-1)^{l-1}}{(2i\pi m)^l} [1 - (-1)^l] \cdot [\rho^{(l-1)}(1) - \rho^{(l-1)}(0)] \\ &\quad + \frac{(-1)^N}{(2i\pi m)^{N-1}} [1 - (-1)^{N-1}] \int_0^1 \rho^{(N)}(x) (e^{2i\pi mx} + e^{-2i\pi mx}) dx. \end{aligned} \quad (72)$$

In the right hand side of (72) zero terms of the first sum correspond to even l . Then, taking $l = 2s + 1$ and replacing $N \rightarrow 2N + 2$, from (72) we obtain

$$\begin{aligned} \int_0^1 (e^{2i\pi mx} - e^{-2i\pi mx}) \rho(x) dx &= 2 \sum_{s=0}^N \frac{1}{(2i\pi n)^{2s+1}} [\rho^{(2s)}(1) - \rho^{(2s)}(0)] \\ &\quad + \frac{2}{(2i\pi n)^{2N+1}} \int_0^1 \rho^{(2N+2)}(x) (e^{2i\pi nx} + e^{-2i\pi nx}) dx. \end{aligned} \quad (73)$$

Then, according to (70), (71) we obtain at $m = n2^k$ the expression for Σ_1 :

$$\begin{aligned} \Sigma_1 &= \sum_{k=0}^{\infty} \frac{1}{z^{k+1}} \sum_{n \neq 0} \sum_{l=1}^N \frac{e^{2i\pi nx}}{(2i\pi n)^{2l} 2^{2kl}} [\rho^{(2l-1)}(1) - \rho^{(2l-1)}(0)] \\ &\quad + \sum_{k=0}^{\infty} \frac{1}{z^{k+1}} \sum_{n \neq 0} \frac{e^{2i\pi nx}}{(2i\pi n)^{2N} 2^{2Nk}} \times \int_0^1 \rho^{(2N+1)}(x') (e^{2i\pi n 2^k x'} + e^{-2i\pi n 2^k x'}) dx'. \end{aligned} \quad (74)$$

Now we restrict our consideration by the set of all densities $\rho(x)$ such that the series in the right hand side of (74) converge when $N \rightarrow \infty$. In the domain $z > 1/2$ this condition is the condition of the convergence of the series:

$$\sum_l (2i\pi)^{-2l} [\rho^{(2l-1)}(1) - \rho^{(2l-1)}(0)] < \infty. \quad (75)$$

The condition

$$(2\pi)^{-2N} \int_0^1 |\rho^{(2N-1)}(x)| dx \rightarrow 0 \quad \text{at} \quad N \rightarrow \infty \quad (76)$$

guaranties that the latter term in the right hand side of (74) including the integral converges to zero at $N \rightarrow \infty$. So, keeping the condition (75), (76) in the limit when $N \rightarrow \infty$ we obtain for the Σ_1 the expression

$$\Sigma_1 = \frac{1}{z} \sum_{k=0}^{\infty} \frac{1}{z^k} \sum_{n \neq 0} e^{2i\pi nx} \cdot \sum_{l=1}^{\infty} \frac{1}{(2i\pi n)^{2l} 2^{2lk}} [\rho^{(2l-1)}(1) - \rho^{(2l-1)}(0)]. \quad (77)$$

If $\left| \frac{1}{z^{2^{2l}}} \right| < 1$ i.e. $|z^{2^{2l}}| > 1$ for any l , $l \geq 1$, then the following is valid:

$$\frac{1}{z} \sum_{k=0}^{\infty} \frac{1}{(z^{2^{2l}})^k} = \frac{1}{z} \left(\frac{1}{1 - \frac{1}{z^{2^{2l}}}} \right) = \frac{1}{z} \cdot \frac{z}{z - \frac{1}{z^{2^{2l}}}} = \frac{1}{z - \frac{1}{z^{2^{2l}}}}.$$

So, if $|z| > 1/4$ we obtain from (77) and (78):

$$\Sigma_1 = \sum_{l=1}^{\infty} \frac{[\rho^{(2l-1)}(1) - \rho^{(2l-1)}(0)]}{z - (1/2)^{2l}} \sum_{n \neq 0} \frac{e^{2i\pi n x}}{(2i\pi n)^{2l}}. \quad (78)$$

We reconstruct the expression (78) in the following way:

$$\sum_{n \neq 0} \frac{e^{2i\pi n x}}{(2i\pi n)^{2l}} = \sum_{n=1}^{\infty} \frac{2 \cos(2\pi n x)}{(2\pi n)^{2l} (-1)^l}. \quad (79)$$

Let us use now the formula for the Bernoulli polynomials [5]:

$$B_{2l}(x) = \frac{(-1)^{l-1} 2(2l)!}{(2\pi)^{2l}} \sum_{n=1}^{\infty} \frac{\cos(2\pi n x)}{n^{2l}}. \quad (80)$$

Then, the equation (79) can be written as

$$\sum_{n \neq 0} \frac{e^{2i\pi n x}}{2i\pi n^{2l}} = -\frac{B_{2l}}{(2l)!}. \quad (81)$$

Taking this into the account we obtain for the Σ_1 the following representation

$$\Sigma_1 = \sum_{l=1}^{\infty} \frac{[\rho^{(2l-1)}(1) - \rho^{(2l-1)}(0)]}{(2l)!} \sum_{n \neq 0} \frac{B_{2l}}{z - (1/2)^{2l}}. \quad (82)$$

Let us consider now the contribution of Σ_2 (see (66)) into the resolvent (64). For the calculation Σ_2 we use the formula (73) and obtain

$$\begin{aligned} \Sigma_2 &= \frac{1}{z} \sum_{n \neq 0} e^{2i\pi n x} \sum_{l=0}^N \frac{1}{(2i\pi n)^{2l+1}} [\rho^{(2l)}(1) - \rho^{(2l)}(0)] \\ &\quad + \frac{1}{z} \sum_{n \neq 0} \frac{e^{2i\pi n x}}{(2i\pi n)^{2N+1}} \cdot \int_0^1 \rho^{(2N+2)}(x') (e^{2i\pi n x'} + e^{-2i\pi n x'}) dx'. \end{aligned} \quad (83)$$

At the same conditions (75), (76) one can consider a limit at $N \rightarrow \infty$ and as a result we have

$$\Sigma_2 = \frac{1}{z} \sum_{l=0}^{\infty} [\rho^{(2l)}(1) - \rho^{(2l)}(0)] \sum_{n \neq 0} \frac{e^{2i\pi n x}}{(2i\pi n)^{2l+1}}. \quad (84)$$

We rewrite the inner sum in (84) as follows:

$$\sum_{n \neq 0} \frac{e^{2i\pi n x}}{(2i\pi n)^{2l+1}} = \sum_{n=1}^{\infty} \frac{e^{2i\pi n x} - e^{-2i\pi n x}}{(2\pi n)^{2l+1} (i)^{2l+1}} = \sum_{n=1}^{\infty} \frac{(-1)^l 2 \sin(2\pi n x)}{(2\pi n)^{2l+1}}. \quad (85)$$

Once again we use the formula for the Bernoulli polynomials [5]:

$$B_{2l+1}(x) = \frac{(-1)^{l+1} 2(2l+1)!}{(2\pi)^{2l+1}} \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{n^{2l+1}}, \quad (86)$$

and find that (85) can be written in the form

$$\sum_{n \neq 0} \frac{e^{2i\pi nx}}{(2i\pi n)^{2l+1}} = -\frac{B_{2l+1}}{(2l+1)!}. \quad (87)$$

Then, for the contribution of Σ_2 we have

$$\Sigma_2 = -\frac{1}{z} \sum_{l=1}^{\infty} \frac{[\rho^{(2l)}(1) - \rho^{(2l)}(0)]}{(2l+1)!} B_{2l+1}(x). \quad (88)$$

Now let us calculate Σ_3 . For this purpose we substitute (71) into (67) at $m = (2l+1)2^k$:

$$\begin{aligned} \Sigma_3 &= \sum_{k \geq 0} \frac{i}{z^2 z^k \pi} \sum_{l, n; n \neq 0} \frac{e^{2i\pi nx}}{2l - 2n + 1} \sum_{s=1}^N \frac{\rho^{(2s-1)}(1) - \rho^{(2s-1)}(0)}{[2i\pi(2l+1)]^{2s} 2^{2ks}} \\ &\quad - \sum_{k=0}^{\infty} \frac{2i}{\pi(2i\pi 2^k)^{2N}} \sum_{l, n; n \neq 0} \frac{e^{2i\pi nx}}{(2l+1)^{2N} (2l - 2n + 1)} \times \\ &\quad \times \int_0^1 \rho^{(2N+1)}(x) \left(e^{2i\pi(2l+1)2^k x} + e^{-2i\pi(2l+1)2^k x} \right) dx. \end{aligned} \quad (89)$$

Since the series over n in the equation (89) does not converges absolutely one can not take the limit at $N \rightarrow \infty$ in (89). Therefore in the first term in (89) we change the places of the last finite sum which depends on n and the series which depend on l, n . Then, the first term in (89) has the form

$$\sum_{k \geq 0} \frac{2}{iz^2 z^k} \sum_{s=1}^N \frac{\rho^{(2s-1)}(1) - \rho^{(2s-1)}(0)}{[(2i\pi)^{2s} 2^{2ks}]} \sum_{\substack{l, n \\ n \neq 0}} \frac{e^{2i\pi nx}}{[2n - (2l+1)] (2l+1)^s \pi}. \quad (90)$$

We consider the series which depend on n in (90) and transform them in the following way:

$$\sum_{n \neq 0} \frac{e^{2i\pi nx}}{[2n - (2l+1)] \pi} = \sum_n \frac{e^{2i\pi nx}}{[2n - M] \pi} + \frac{1}{(2l+1)\pi} = \sum_n \frac{e^{2i\pi nx}}{2n\pi - M\pi} + \frac{1}{M\pi}, \quad (91)$$

where $M = 2l+1$. We use now the Poisson formula for the summation of the series [30]. Let $g(x)$ be a function of the bounded variation on the interval $(-\infty, +\infty)$ such that the integral in the Poisson summation formula converges (for example in the sense of the principal value). Then, the Poisson summation formula has a form

$$\sum_{n=-\infty}^{\infty} g(2\pi n) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{+\infty} g(t) e^{-itn} dt. \quad (92)$$

Now we use equation (92) for summation of the series (91). In our case

$$g(t) = \frac{e^{itx}}{t - M\pi}, \quad (93)$$

and hence from (91) we get

$$\sum_n \frac{e^{2i\pi nx}}{2\pi n - \pi M} = \sum_n \frac{1}{2\pi} \text{V.p.} \int_{-\infty}^{+\infty} \frac{e^{it(x-n)}}{t - \pi M} dt. \quad (94)$$

Let us denote $\alpha = x - n$. Then, depending on the sign of α one should calculate the integral in (94) by residues closing the contour into the upper ($\alpha > 0$) or into the lower ($\alpha < 0$) halfplane. Consider the contour Γ_R consisting of two intervals of the real axis: $[-R, -\varepsilon]$ and $[\varepsilon, R]$, $\varepsilon > 0$; the halfcircumference C_ε with the radius $\varepsilon > 0$ lying in the upper complex halfplane and of the halfcircumference C_R with the radius $R > \varepsilon > 0$ lying also there. Then (the direction of the clockwise or contra clockwise turning is noted by arrows):

$$\int_{\Gamma_R} = \int_{-R}^{-\varepsilon} + \int_{\varepsilon}^R + \int_{\rightarrow C_\varepsilon} + \int_{\leftarrow C_R}. \quad (95)$$

One could show that

$$\int_{\leftarrow C_R} \rightarrow 0 \quad \text{if} \quad R \rightarrow \infty, \quad (96)$$

and

$$\lim_{\varepsilon \rightarrow 0} \left(\int_{\rightarrow C_\varepsilon} \right) = -i\pi \text{res} \left\{ \frac{e^{it\alpha}}{t - \pi M} \right\}_{t=\pi M}. \quad (97)$$

By means of (95) – (97) we obtain for $\alpha > 0$:

$$\text{V.p.} \int_{-\infty}^{+\infty} \frac{e^{it\alpha}}{t - \pi M} dt = i\pi \text{res} \left\{ \frac{e^{it\alpha}}{t - \pi M} \right\}_{t=\pi M} = -i\pi e^{i\pi M\alpha}. \quad (98)$$

Quite similar for $\alpha < 0$, closing the contour into the low halfplane, we obtain

$$\text{V.p.} \int_{-\infty}^{+\infty} \frac{e^{it\alpha}}{t - \pi M} dt = -i\pi \text{res} \left\{ \frac{e^{it\alpha}}{t - \pi M} \right\}_{t=\pi M} = -i\pi e^{i\pi M\alpha}. \quad (99)$$

Turning back to the series (91) one can see that these series are splitted into two series in depending on the sign of $x - n$, $x \in (0, 1)$. Namely

$$x - n > 0 \iff x > n \iff n = 0, -1, -2, \dots \quad (100)$$

$$x - n < 0 \iff x < n \iff n = 1, 2, 3, \dots \quad (101)$$

Hence

$$\begin{aligned} \sum_n \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{it(x-n)}}{t - \pi M} dt &= \frac{i}{2} \sum_{n=-\infty}^0 e^{i\pi M(x-n)} - \frac{i}{2} \sum_{n=1}^{\infty} e^{i\pi M(x-n)} = \\ &= \frac{i}{2} e^{i\pi Mx} + \frac{i}{2} \sum_{n=1}^{\infty} e^{i\pi Mx} e^{i\pi Mn} - \frac{i}{2} \sum_{n=1}^{\infty} e^{i\pi Mx} e^{-i\pi Mn}. \end{aligned} \quad (102)$$

Since for every M, n we have

$$e^{-i\pi Mn} = e^{i\pi Mn} = \begin{cases} +1, & Mn = 2s \\ -1, & Mn = 2s + 1. \end{cases} \quad (103)$$

It means that two latter series in (102) are cancelled, and we obtain

$$\sum_n \frac{e^{2i\pi nx}}{2\pi n - \pi M} = \sum_n \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{it(x-n)}}{t - \pi M} dt = \frac{i}{2} e^{i\pi M x}, \quad M = 2l + 1. \quad (104)$$

However in (104) we are interested in the series of the left hand side not for all n , but only for $n \neq 0$. So, from (91) it follows that

$$\sum_{n \neq 0} \frac{e^{2i\pi nx}}{2\pi n - \pi M} = \sum_n \frac{e^{2i\pi nx}}{2\pi n - \pi M} + \frac{1}{\pi(2l + 1)} = \frac{i}{2} e^{i\pi(2l+1)x} + \frac{1}{\pi(2l + 1)}. \quad (105)$$

We substitute now this expression in the series depended of l in (90) and obtain

$$\sum_{\substack{l, n \\ n \neq 0}} \frac{e^{2i\pi nx}}{(2\pi n - (2l + 1)\pi)(2l + 1)^{2s}} = \sum_l \frac{i}{2} \frac{e^{i\pi(2l+1)x}}{(2l + 1)^{2s}} + \sum_l \frac{1}{\pi(2l + 1)^{2s+1}}. \quad (106)$$

The latter series in the right hand side in (106) obviously is equals to zero:

$$\sum_l \frac{1}{\pi(2l + 1)^{2s+1}} = 0. \quad (107)$$

Taking into account relations written above let us consider now the limit at $N \rightarrow \infty$ in Σ_3 . At the same conditions as above the terms in Σ_3 including the integrals of $\rho^{2N+1}(x)$ tend to zero and so in the limit $N \rightarrow 0$ we get

$$\Sigma_3 = \sum_{k=0} \frac{1}{z^2 z^k} \sum_{s=1}^{\infty} \frac{[\rho^{(2s-1)}(1) - \rho^{(2s-1)}(0)]}{(2i\pi)^{2s} 2^{2ks}} \sum_l \frac{e^{i\pi(2l+1)x}}{(2l + 1)^{2s}}. \quad (108)$$

We use the fact that if $|z| > 1/4$ (see (78)) then

$$\sum_{k=0}^{\infty} \frac{1}{(z 2^{2s})^k} = \frac{z}{z - (1/2)^{2s}} \quad (109)$$

and for the contribution of Σ_3 in the resolvent we obtain the following representation:

$$\begin{aligned} \Sigma_3 &= \sum_{s=1}^{\infty} \frac{[\rho^{(2s-1)}(1) - \rho^{(2s-1)}(0)]}{z \left(z - \frac{1}{2^{2s}}\right) (2i\pi)^{2s}} \sum_l \frac{e^{i\pi(2l+1)x}}{(2l + 1)^{2s}} = \\ &= \sum_{s=1}^{\infty} \frac{[\rho^{(2s-1)}(1) - \rho^{(2s-1)}(0)]}{z \left(z - \frac{1}{2^{2s}}\right) (-1)^s (2\pi)^{2s}} \sum_{l=0}^{\infty} \frac{e^{i\pi(2l+1)x} + e^{-i\pi(2l+1)x}}{(2l + 1)^{2s}} = \\ &= \sum_{s=1}^{\infty} \frac{2 [\rho^{(2s-1)}(1) - \rho^{(2s-1)}(0)]}{z \left(z - \frac{1}{2^{2s}}\right) 2^{2s} \pi^{2s}} \sum_{l=0}^{\infty} \frac{\cos(2l + 1)\pi x}{(2l + 1)^{2s}}. \end{aligned} \quad (110)$$

Now we use the well-known [5] Euler polynomials representation:

$$E_{2s-1}(x) = \frac{4(-1)^s (2s - 1)!}{\pi^{2s}} \sum_{l=0}^{\infty} \frac{\cos(2l + 1)\pi x}{(2l + 1)^{2s}} \quad (111)$$

and obtain for Σ_3 :

$$\Sigma_3 = \sum_{s=1}^{\infty} \frac{[\rho^{(2s-1)}(1) - \rho^{(2s-1)}(0)]}{z \left(z - \frac{1}{2^{2s}}\right) 2^{2s}} \frac{E_{2s-1}(x)}{2(2s-1)!}. \quad (112)$$

To get the desired expression for Σ_3 we shall transform the denominator in (112). Namely, since

$$\frac{1}{z \left(z - \frac{1}{2^{2s}}\right)} = \left(\frac{1}{z - \frac{1}{2^{2s}}} - \frac{1}{z} \right) 2^{2s}, \quad (113)$$

and hence

$$\frac{1}{z \left(1 - \frac{1}{2^{2s}}\right) 2^{2s}} = \frac{1}{z - \frac{1}{2^{2s}}} - \frac{1}{z} \quad (114)$$

we finally obtain for Σ_3

$$\Sigma_3 = \sum_{s=1}^{\infty} \frac{[\rho^{(2s-1)}(1) - \rho^{(2s-1)}(0)]}{\left(z - \frac{1}{2^{2s}}\right) (2s-1)!} \cdot \frac{E_{2s-1}(x)}{2} - \frac{1}{z} \sum_{s=1}^{\infty} \frac{[\rho^{(2s-1)}(1) - \rho^{(2s-1)}(0)]}{(2s-1)!} \cdot \frac{E_{2s-1}(x)}{2}. \quad (115)$$

Now we return again to the presentation (64) of the resolvent. As a result of the above calculations we find the expressions for the contributions of Σ_1 , Σ_2 , Σ_3 in the terms of the Bernoulli and Euler polynomials. We sum this contributions according to (64) and collect the terms at the different peculiarities by z . Namely collecting terms at $1/z$ according (115), (88) we obtain

$$\begin{aligned} & -\frac{1}{z} \sum_{l=0}^{\infty} \frac{[\rho^{(2l)}(1) - \rho^{(2l)}(0)]}{(2l+1)!} B_{2l+1}(x) - \frac{1}{z} \sum_{l=0}^{\infty} \frac{[\rho^{(2l+1)}(1) - \rho^{(2l+1)}(0)]}{(2l+1)!(2l+2)} \frac{E_{2l-1}(x)(2l+2)}{2} = \\ & = -\frac{1}{z} \sum_{l=0}^{\infty} \frac{[\rho^{(2l)}(0) - \rho^{(2l)}(1)]}{(2l+1)!} B_{2l+1}(x) + \frac{1}{z} \sum_{l=0}^{\infty} \frac{[\rho^{(2l+1)}(0) - \rho^{(2l+1)}(1)]}{(2l+2)!} (l+1) E_{2l+1}(x) = \\ & = -\frac{1}{z} \sum_{m=0}^{\infty} \frac{[\rho^{(m)}(1) - \rho^{(m)}(0)]}{(m+1)!} \psi_m(x). \end{aligned} \quad (116)$$

Here

$$\psi_m(x) = \begin{cases} B_{2l+1}(x), & m = 2l \\ (l+1)E_{2l+1}(x), & m = 2l+1. \end{cases} \quad (117)$$

We collect now the terms at the pole peculiarities $\left(z - \frac{1}{2^{2s}}\right)^{-1}$ in the resolvent (64) using (112), (82):

$$\begin{aligned} & \sum_{l=1}^{\infty} \frac{[\rho^{(2l-1)}(0) - \rho^{(2l-1)}(1)]}{z - \frac{1}{2^{2l}}} \cdot \left\{ \frac{B_{2l}(x)}{(2l)!} - \frac{E_{2l-1}(x)}{2(2l-1)!} \right\} = \\ & = \sum_{l=1}^{\infty} \frac{[\rho^{(2l-1)}(1) - \rho^{(2l-1)}(0)]}{z - \frac{1}{2^{2l}}} \varphi_{2l-1}(x), \end{aligned} \quad (118)$$

where

$$\varphi_{2l-1}(x) = -lE_{2l-1}(x) + B_{2l}(x). \quad (119)$$

From (116) – (119) according to (64) we obtain the action of the resolvent $R(z)$ in the form

$$\begin{aligned} (R(z)\rho)(x) &= \frac{1}{1-z} \int_0^1 \rho(x') dx' \mathbf{1}(x) + \sum_{l=1}^{\infty} \frac{[\rho^{(2l-1)}(0) - \rho^{(2l-1)}(1)]}{\left(z - \frac{1}{2^{2l}}\right) (2l)!} \varphi_{2l-1}(x) \\ &+ \frac{1}{z} \sum_{m=0}^{\infty} \frac{[\rho^{(m)}(0) - \rho^{(m)}(1)]}{(m+1)!} \psi_m(x). \end{aligned} \quad (120)$$

This formula (as one for the Renyi map [1], [4]) determine the continuation $R^c(z)$ of the resolvent $R(z)$ of the operator U_T to the whole complex plane \mathbf{C} of the spectral parameter z . The continuation $R^c(z)$ needs the restriction U_T to the suitable functional space $\Phi \subset \mathcal{A}[0, 1]$. In the examining case the choice of Φ is caused by the reasons of the rightness of the developing above calculations, i. e. the functions $\rho(x)$, $\rho \in \Phi$ should satisfy the conditions (75), (76). As for the Renyi map [1], [4], the detailed description of the test space can be found in [16].

It follows from (120) that the continuation $R^c(z)$ of the restricted resolvent $R(z)|_{\Phi}$ can be written in the form

$$R^c(z) = \frac{1}{1-z} |\mathbf{1}\rangle\langle\mathbf{1}| + \frac{1}{z} \sum_{m=0}^{\infty} |\psi_m\rangle\langle\tilde{\psi}_m| + \sum_{l=1}^{\infty} \frac{1}{z - \frac{1}{2^{2l}}} |\varphi_{2l-1}\rangle\langle\tilde{\varphi}_{2l-1}|, \quad (121)$$

where the functions ψ_m and φ_{2l-1} are determined by (117) and (119) respectively. By $\langle\tilde{\psi}_m|$ and $\langle\tilde{\varphi}_{2l-1}|$ are denoted functionals over the Φ (i. e. the elements of the dual space Φ^\times) which are acting to $\rho \in \Phi \subset \mathcal{A}[0, 1]$ according to the rules

$$\langle\tilde{\psi}_m|\rho\rangle = \frac{[\rho^{(m)}(0) - \rho^{(m)}(1)]}{(m+1)!}, \quad (122)$$

$$\langle\tilde{\varphi}_{2l-1}|\rho\rangle = - \frac{[\rho^{(2l-1)}(1) - \rho^{(2l-1)}(0)]}{(2l)!}. \quad (123)$$

It follows from these formulas that the functionals $\langle\tilde{\psi}_m|$ and $\langle\tilde{\varphi}_{2l-1}|$ can be represented in the form of the derivatives of δ -functions:

$$\psi_m = (-1)^m \frac{[\delta^{(m)}(x) - \delta^{(m)}(x-1)]}{(m+1)!}, \quad (124)$$

$$\varphi_{2l-1} = - \frac{[\delta^{(2l-1)}(x-1) - \delta^{(2l-1)}(x)]}{(2l)!}, \quad (125)$$

Coming back to the representation(121), it should be checked, that $\varphi_{2l-1}(x)$ satisfy the equation

$$U_T \varphi_{2l-1}(x) = \frac{1}{2^{2l}} \varphi_{2l-1}(x), \quad (126)$$

or, what is the same, the equation

$$\frac{1}{2} \left[\varphi_{2l-1} \left(\frac{x}{2} \right) + \varphi_{2l-1} \left(1 - \frac{x}{2} \right) \right] = \frac{1}{2^{2l}} \varphi_{2l-1}(x), \quad (127)$$

where $\varphi_{2l-1}(x)$ is given by (119). To prove (127) we use the known formulas [5]:

$$B_{2l}\left(\frac{x}{2}\right) = (-1)^{2l} B_{2l}\left(1 - \frac{x}{2}\right) = B_{2l}\left(1 - \frac{x}{2}\right), \quad (128)$$

$$E_{2l-1}\left(\frac{x}{2}\right) = (-1)^{2l-1} E_{2l-1}\left(1 - \frac{x}{2}\right) = -E_{2l-1}\left(1 - \frac{x}{2}\right), \quad (129)$$

$$E_{n-1}(x) = \frac{2}{n} \left\{ B_n(x) - 2^n B_n\left(\frac{x}{2}\right) \right\}. \quad (130)$$

Then, by means of (128) – (130), we have

$$\begin{aligned} \varphi_{2l-1}\left(\frac{x}{2}\right) + \varphi_{2l-1}\left(1 - \frac{x}{2}\right) &= B_{2l}\left(\frac{x}{2}\right) + B_{2l}\left(1 - \frac{x}{2}\right) - lE_{2l-1}\left(\frac{x}{2}\right) - lE_{2l-1}\left(1 - \frac{x}{2}\right) = \\ &= 2B_{2l}\left(\frac{x}{2}\right). \end{aligned} \quad (131)$$

Hence to prove (127), it is necessary to be sure that

$$B_{2l}\left(\frac{x}{2}\right) = \frac{1}{2^{2k}} [B_{2l}(x) - lE_{2l-1}(x)]. \quad (132)$$

It follows from (132) that

$$lE_{2l-1}(x) = B_{2l}(x) - 2^{2l} B_{2l}\left(\frac{x}{2}\right). \quad (133)$$

Substituting this relation to the right hand side of (132) we obtain the identity

$$B_{2l}\left(\frac{x}{2}\right) = \frac{1}{2^{2l}} \left\{ B_{2l}(x) - B_{2l}(x) - 2^{2l} B_{2l}\left(\frac{x}{2}\right) \right\} = B_{2l}\left(\frac{x}{2}\right)$$

which proves (127) and hence (126). It means that $\varphi_{2l-1}(x)$ are the eigenfunctions of the Frobenius–Perron operator $U_T|_{\Phi}$, restricted to the test functions space $\Phi \subset \mathcal{A}[0, 1]$.

Let us mention that $\varphi_{2l-1}(x)$ can be expressed in the terms of even Bernoulli polynomials. Namely using (133) we see that

$$\varphi_{2l-1}(x) = B_{2l}(x) - lE_{2l-1}(x) = 2^{2l} B_{2l}\left(\frac{x}{2}\right). \quad (134)$$

Similarly one can check that the functions $\psi_m(x) \in \Phi \subset \mathcal{A}[0, 1]$, given by the equation (117), are included in $\ker U_T|_{\Phi}$ i. e. satisfy the equation

$$(U_T \psi_m)(x) = 0, \quad m \in \mathbf{Z}. \quad (135)$$

Therefore they are the eigenfunctions of $U_T|_{\Phi}$ corresponding to the eigenvalue $z = 0$ of the infinite degeneracy. Turning again to the (121) we determine the extended resolvent $\tilde{R}_V(z)$ of the Koopman operator V_T for the tent map by duality, which is defined in the Introduction. Then on the base of (121) we have for $\tilde{R}_V(z)$:

$$\tilde{R}_V(z) = \frac{1}{1-z} |\mathbf{1}\rangle\langle\mathbf{1}| + \frac{1}{z} \sum_{m=0}^{\infty} |\tilde{\psi}_m\rangle\langle\psi_m| + \sum_{l=0}^{\infty} \frac{1}{z - \frac{1}{2^{2l}}} |\tilde{\varphi}_{2l-1}\rangle\langle\varphi_{2l-1}|. \quad (136)$$

Now we simplify the notations in (122), (123), taking into account that the functionals $\langle\tilde{\psi}_m|$ and $\langle\tilde{\varphi}_{2l-1}|$ act to $\rho \in \Phi$ equally. Namely, we determine the functional $\langle\tilde{\chi}_m|$ by its action on $\rho \in \Phi$ as follows

$$\langle\tilde{\chi}_m|\rho\rangle = \frac{\rho^{(m)}(1) - \rho^{(m)}(0)}{(m+1)!}.$$

Then

$$\langle \tilde{\psi}_m | = \langle \tilde{\chi}_m |, \quad \langle \tilde{\varphi}_{2l-1} | = \langle \tilde{\chi}_{2l-1} |.$$

Using these notations and (117), (119) we can rewrite (121) in the form:

$$\begin{aligned} -R^c(z) &= \frac{1}{z-1} |\mathbf{1}\rangle \langle \mathbf{1}| + \frac{1}{z} \left(\sum_{l=0}^{\infty} |B_{2l+1}\rangle \langle \tilde{\chi}_{2l}| + \sum_{l=1}^{\infty} |l E_{2l-1}\rangle \langle \tilde{\psi}_{2l-1}| \right) \\ &\quad + \sum_{l=1}^{\infty} \frac{1}{z - \frac{1}{2^l}} |2^{2l} B_{2l} \left(\frac{x}{2} \right)\rangle \langle \tilde{\chi}_{2l-1}|. \end{aligned} \quad (137)$$

We check now the completeness for $R^c(z)$ in the strong sense on the test functions space Φ using the fact [1], [16], [17] that Φ coincides with the space of the entire functions of exponential type less than 2π . For this purpose the following equation should be checked:

$$\rho = - \sum_{l=1}^{\infty} \text{res}_{z=2^{-2l}} R^c \rho - \text{res}_{z=1} R^c \rho - \text{res}_{z=0} R^c \rho \quad \forall \rho \in \Phi, \quad (138)$$

Proceeding from (137), the right hand side of (138) is equals to

$$\begin{aligned} |\mathbf{1}\rangle \langle \mathbf{1}| \rho + \sum_{l=0}^{\infty} |B_{2l+1}\rangle \langle \tilde{\chi}_{2l}| \rho + \sum_{l=1}^{\infty} |l E_{2l-1} + 2^{2l} B_{2l} \left(\frac{x}{2} \right)\rangle \langle \tilde{\psi}_{2l-1}| \rho &= \\ &= |\mathbf{1}\rangle \langle \mathbf{1}| \rho + \sum_{n=1}^{\infty} |B_n\rangle \langle \tilde{\chi}_{n-1}| \rho. \end{aligned} \quad (139)$$

Here the equation (134) is used as well as the summation over even and odd indices. Then (139) can be written also in the form:

$$\int_0^1 \rho(x) dx + \sum_{n=1}^{\infty} \frac{\rho^{(n-1)}(1) - \rho^{(n-1)}(0)}{n!} B_n(x) = \rho(x). \quad (140)$$

Here the equality of the left hand side to the function $\rho \in \Phi$ follows from the Euler–Maclaren formula (see [17]) for example). Hence, the completeness for $R^c(z)$ and so for the extension of the Koopman operator $\tilde{R}_V(z)$ is proved.

It should be noted, that the property of the completeness can be written formally in the operator form:

$$\mathbf{I} = |\mathbf{1}\rangle \langle \mathbf{1}| + \sum_{n=1}^{\infty} |B_n\rangle \langle \tilde{\chi}_{n-1}|. \quad (141)$$

Let us introduce the operators

$$\Pi_n = |B_n\rangle \langle \tilde{\chi}_{n-1}| \quad (142)$$

on $\Phi \subset \mathcal{A}[0, 1]$ and check that Π_n satisfies the following conditions

$$\Pi_l \Pi_m = \delta_{lm} \Pi_m, \quad \Pi_0 \Pi_m = 0, \quad (143)$$

where Π_0 is determined above by (62). The property (143) means that $\{\Pi_n\}$ is the set of the orthoprojections. We shall use the fact that the Bernoulli polynomials belong to the family of the Appel polynomials [5], [17] and hence the following equation is valid:

$$B'_n = n B_{n-1}(x) \quad (144)$$

(here the prime means the derivative by x). From (144) it follows that

$$B_m^{(l-1)}(x) = \frac{m!}{(m-l+1)!} B_{m-l+1}(x). \quad (145)$$

It is well known [5] also that

$$\begin{aligned} B_{2n}(0) &= B_{2n}(1), \\ B_{2n+1}(0) &= B_{2n+1}(1) = 0 \quad n \geq 1, \\ B_1(0) &= -1/2, \\ B_1(1) &= 1/2. \end{aligned} \quad (146)$$

On the base of (145) and (146) we conclude that the equation

$$\frac{B_m^{(l-1)}(1) - B_m^{(l-1)}(0)}{l!} = \delta_{lm} [B_1(1) - B_1(0)] = \delta_{lm} \quad (147)$$

is valid. It should be noted that

$$\langle \tilde{\chi}_{l-1} | B_m \rangle = \frac{B_m^{(l-1)}(1) - B_m^{(l-1)}(0)}{l!} = \delta_{lm}. \quad (148)$$

Then

$$\Pi_l \Pi_m = |B_l\rangle \langle \tilde{\chi}_{l-1} | B_m \rangle \langle \tilde{\chi}_{m-1} | = \delta_{lm} \Pi_m,$$

and so the first of the equations (143) is proved. Now we should check the second equation in (143) i. e. we should calculate

$$\Pi_0 \Pi_m = |\mathbf{1}\rangle \langle \mathbf{1} | B_m \rangle \langle \tilde{\chi}_{m-1} |. \quad (149)$$

Again using (146) we find

$$\langle \mathbf{1} | B_m \rangle = \int_0^1 B_n(x) dx = \frac{B_{m+1}(1) - B_{m+1}(0)}{(m+1)!} = 0, \quad (150)$$

and hence the second equation is proved.

Now let us act by the operator U_T to the decomposition (141) from the left. Then taking into account the properties

$$\begin{aligned} \tilde{U}_T |B_{2l+1}\rangle &= 0, \\ \tilde{U}_T |B_{2l}\rangle &= \frac{1}{2^{2l}} |B_{2l} - lE_{2l-1}\rangle, \\ \tilde{U}_T |\mathbf{1}\rangle &= |\mathbf{1}\rangle \end{aligned} \quad (151)$$

which follow from (131), (132), (135), we obtain the generalized spectral decomposition of the Frobenius–Perron operator \tilde{U}_T corresponding to the extended resolvent $R^c(z)$:

$$\tilde{U}_T = |\mathbf{1}\rangle \langle \mathbf{1}| + \sum_{l=1}^{\infty} \frac{1}{2^{2l}} |B_{2l} - lE_{2l-1}\rangle \langle \tilde{\chi}_{2l-1}|. \quad (152)$$

This is the main result of the work. It can be rewritten using the projections which are different from ones in (142). Namely let us determine the operators

$$\hat{\Pi}_l = |B_{2l} - lE_{2l+1}\rangle \langle \tilde{\chi}_{2l-1}| \quad (153)$$

and rewrite (152) in the form

$$\tilde{U}_T = \Pi_0 + \sum_{l=1}^{\infty} \frac{1}{2^{2l}} \hat{\Pi}_l. \quad (154)$$

We shall check that operators $\hat{\Pi}_l$ are the orthoprojections again, i. e.

$$\hat{\Pi}_l \hat{\Pi}_m = \delta_{lm} \hat{\Pi}_m, \quad \Pi_0 \hat{\Pi}_l = 0. \quad (155)$$

Indeed from (36) it follows that

$$\hat{\Pi}_l = 2^{2l} |B_{2l} \left(\frac{x}{2} \right) \rangle \langle \tilde{\chi}_{2l-1} |, \quad (156)$$

and in order to prove (155) it is enough to verify that

$$\langle \tilde{\chi}_{2l-1} | B_{2m} \left(\frac{x}{2} \right) \rangle = \delta_{lm} \frac{1}{2^{2l}}. \quad (157)$$

The latter is checked by the direct calculation:

$$\begin{aligned} \langle \tilde{\chi}_{2l-1} | B_{2m} \left(\frac{x}{2} \right) \rangle &= \frac{1}{2^{2l}} \frac{B_{2m}^{(2l-1)}(1/2) - B_{2m}^{(2l-1)}(0)}{(2l)!} = \frac{2m!}{2^{2l}} \cdot \frac{B_{2m-2l+1}(1/2) - B_{2m-2l+1}(0)}{(2l)!(2m-2l+1)!} = \\ &= \frac{1}{2^{2l}} \delta_{lm} \left[B_1 \left(\frac{1}{2} \right) - B_1(0) \right] = \frac{1}{2^{2l}} \delta_{lm}, \end{aligned} \quad (158)$$

where we use the properties of the Bernoulli polynomials:

$$\begin{aligned} B_{2n+1}(0) &= B_{2n+1}(1/2) = 0, \quad n \geq 1 \\ B_1(0) &= -1/2 \\ B_1(1/2) &= 0 \end{aligned} \quad (159)$$

The second equation in (155) is also can be obtained directly by using the properties of the Euler polynomials:

$$\Pi_0 \hat{\Pi}_l = |\mathbf{1}\rangle \langle \mathbf{1} | B_{2l} - l E_{2l-1} \rangle \langle \tilde{\chi}_{2l-1} |. \quad (160)$$

According to (150) $\langle \mathbf{1} | B_m \rangle = 0$ for all m and so we have to calculate

$$\langle \mathbf{1} | E_{2l-1} \rangle = \int_0^1 E_{2l-1}(x) dx = \frac{E_{2l}(1) - E_{2l}(0)}{(2l)!}. \quad (161)$$

Using the fact that $E_{2l}(1) = E_{2l}(0)$ we get sure that $\Pi_0 \cdot \hat{\Pi}_l = 0$.

We have proved above (see (126), (127)) that the original operator U_T acts to the right eigenfunctions "correctly". Now we should check that the extended operator \tilde{U}_T given by the generalized spectral decomposition (154) in the Gelfand triplet $\Phi \subset L_2(0, 1) \subset \Phi^\times$ acts in its spectral representation in the following way:

$$\begin{aligned} \tilde{U}_T |B_{2m} \left(\frac{x}{2} \right) \rangle &= \frac{1}{2^{2m}} |B_{2m} \left(\frac{x}{2} \right) \rangle, & \tilde{U}_T |\mathbf{1}\rangle &= |\mathbf{1}\rangle, \\ \tilde{U}_T |B_{2l+1}\rangle &= 0, & \tilde{U}_T |E_{2l-1}\rangle &= 0. \end{aligned}$$

Indeed from (156), (157) it follows that

$$\hat{\Pi}_l |B_{2m} \left(\frac{x}{2} \right) \rangle = \delta_{ml} \frac{1}{2^{2m}} |B_{2m} \left(\frac{x}{2} \right) \rangle, \quad (162)$$

and from (160), (161) it follows that

$$\Pi_0 |B_{2m} \left(\frac{x}{2} \right) \rangle = 0, \quad \Pi_0 |E_{2l-1}(x) \rangle = 0. \quad (163)$$

Consequently from the equation (150) one can obtain

$$\Pi_0 |B_{2m+1}(x) \rangle = 0, \quad (164)$$

$$\hat{\Pi}_l |B_{2m-1}(x) \rangle = 0. \quad (165)$$

Finally from the fact [17] that the Euler polynomials belong to the set of the Appel polynomials also (see 145) and from the property $E_{2n}(1) = E_{2n}(0)$ it follows that

$$\hat{\Pi}_l |E_{2m-1}(x) \rangle = 0. \quad (166)$$

Collecting the formulas (162)–(166) we obtain (5).

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